Completeness theorem – Henkin model

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Let $\Gamma^* \subset F(\sigma)$ be a maximal consistent theory with the Henkin witness property: for every σ-formula φ, if $\Gamma^* \vdash \exists x\ \varphi$ then $\Gamma^* \vdash \varphi(x/c)$ for some constant symbol c. It can be seen that for every φ, $\Gamma^* \vdash \varphi$ whenever $\varphi \in \Gamma^*$. Let A be the set of $\approx$-equivalence classes of the constant symbols of σ for the following equivalence relation: $c \approx d$ whenever $\Gamma^* \vdash c = d$. Let $\mathfrak{A}$ be a σ-structure with its universe A, with the following interpretations: $(c)_\mathfrak{A} = [c]$; $f_\mathfrak{A}([c]) = d$ iff $\Gamma^* \vdash f(c) = d$; $r_\mathfrak{A}(c)$ iff $\Gamma^* \vdash r(c)$. Note that f and R are compatible with $\approx$: by the equality axioms, $c \approx d$ implies that $\Gamma^* \vdash f(c) = f(d)$ – similarly for R –, thus the value of functions and relations are independent of the equivalence class representatives. Also, by the Henkin property, functions have a value for every argument; i.e. the definitions are good.

In this paper I will prove that $\mathfrak{A}$ is a model of Γ^*.

Notation. Throughout the article, the letters c and d denote constant symbols in the signature σ, e is a valuation over A, f is a function symbol, R is a relation symbol, t is a σ-term, x, y are variables, σ is a signature, φ, ψ are σ-formulas. $[c]$ stands for the equivalence class of c (see below). $\vec{x}$ stands for $x_1, \ldots, x_k$. (Similarly $\vec{c}$ or $\vec{[c]}$). $\varphi(x/c)$ or $t(x/c)$ denotes the formula or term obtained by substituting c for x. (Note that the term c never contains a bound variable of φ.) $\mathfrak{A} \vDash \varphi[e(x/a)]$ iff φ is true in $\mathfrak{A}$ under the valuation e when x is mapped to $a \in A$.

Lemma 1. *For every $\varphi \in F(\sigma)$ with k free variables, $\Gamma^* \vdash \varphi$ if and only if for any $c_i \in \sigma$ constant symbol $\Gamma^* \vdash \varphi(\vec{x}/\vec{c})$.*

Proof. The *only if* part is trivial. For the *if* part, we prove the contrapositive. $\Gamma^* \nvdash \varphi$ iff $\Gamma^* \nvdash \forall \vec{x}\ \varphi$. Then by the maximality of Γ^*: $\Gamma^* \nvdash \forall \vec{x}\ \varphi$. Changing notation: $\Gamma^* \vdash \exists \vec{x}\ \neg\varphi$. By the Henkin witness property, there are $c_1, \ldots, c_k$ constant symbols in σ such that $\Gamma^* \vdash \neg\varphi(x_1/c_1, \ldots, x_k/c_k)$. The consistency of Γ^* ensures that $\Gamma^* \nvdash \varphi(x_1/c_1, \ldots, x_k/c_k)$. □

Lemma 2. *For a σ-term t with k free variables, $\Gamma^* \vdash d = t(\vec{x}/\vec{c}$ for some constant symbol c and d if and only if $[d] = \big(t(\vec{x}/\vec{c})\big)_{\mathfrak{A}}$.*

Proof. Using the Henkin property of Γ^* and the definition of $\mathfrak{A}$, this lemma is easily proved by induction on the construction of t. $\square$

Proposition. *For every $\varphi \in F(\sigma)$, $\Gamma^* \vdash \varphi$ (i.e. $\varphi \in \Gamma^*$) if and only if $\mathfrak{A} \vDash \varphi$.*

Proof. We prove this claim by induction on the construction of φ. Throughout the proof we assume φ has k free variables, $x_1, \dots, x_k$.

Case 1. φ has the form $t_1 = t_2$.

$$\begin{array}{ll}
 & \Gamma^* \vdash t_1 = t_2 \\
\text{iff} & \text{for all } c_i \in \sigma\text{:} \quad \Gamma^* \vdash t_1 = t_2 \ (x_1/c_1 \dots x_k/c_k) \quad \text{(by Lemma 1)} \\
\text{iff} & \text{for all } c_i \in \sigma\text{, some } d \in \sigma\text{, for } j = 1,2\text{:} \\
 & \qquad \Gamma^* \vdash t_j = d \ (\vec{x}/\vec{c}) \quad \text{(by the Henkin property and the equality axioms)} \\
 & \qquad \big(t_j(\vec{x}/\vec{c})\big)_{\mathfrak{A}} = [d] \quad \text{(by Lemma 2)} \\
 & \qquad \big(t_1(\vec{x}/\vec{c})\big)_{\mathfrak{A}} = \big(t_2(\vec{x}/\vec{c})\big)_{\mathfrak{A}} \\
\text{iff} & \text{for all } c_i \in \sigma\text{:} \quad \mathfrak{A} \vDash t_1 = t_2 \ (\vec{x}/\vec{c}) \\
\text{iff} & \text{for all } c_i \in \sigma\text{:} \quad \mathfrak{A} \vDash t_1 = t_2 \ [e(\vec{x}/\vec{[c]})] \quad \text{(by the substitution theorem)} \\
\text{iff} & \mathfrak{A} \vDash \forall x_1 \dots \forall x_k \ t_1 = t_2 \quad \text{(because } A = \{[c] : c \in \sigma\}\text{)} \\
\text{iff} & \mathfrak{A} \vDash t_1 = t_2
\end{array}$$

Case 2. φ has the form $R(t_1, \dots, t_n)$.

$$\begin{array}{ll}
 & \Gamma^* \vdash R(t_1, \dots, t_n) \\
\text{iff} & \text{for all } c_i \in \sigma\text{:} \quad \Gamma^* \vdash R(t_1, \dots, t_n) \ (\vec{x}/\vec{c}) \quad \text{(by Lemma 1)} \\
\text{iff} & \text{for all } c_i \in \sigma\text{, some } d_j \in \sigma \ (j = 1 \dots n)\text{:} \\
 & \qquad \Gamma^* \vdash t_j = d_j \ (\vec{x}/\vec{c}) \quad \text{(by the Henkin property)} \\
 & \qquad \Gamma^* \vdash R(d_1, \dots, d_n) \quad \text{(by the equality axioms)} \\
 & \qquad \mathfrak{A} \vDash R(d_1, \dots, d_n) \quad \text{(by the definition of } \mathfrak{A}\text{)} \\
\text{iff} & \text{for all } c_i \in \sigma\text{:} \quad \mathfrak{A} \vDash R(t_1, \dots, t_n) \ (\vec{x}/\vec{c}) \quad \text{(by Lemma 2)} \\
\text{iff} & \text{for all } c_i \in \sigma\text{:} \quad \mathfrak{A} \vDash R(t_1, \dots, t_n) \ [e(\vec{x}/\vec{[c]})] \quad \text{(by the substitution theorem)} \\
\text{iff} & \mathfrak{A} \vDash \forall x_1 \dots \forall x_k \ R(t_1, \dots, t_n)
\end{array}$$

Case 3. $\varphi \equiv \neg\psi$.

$$
\begin{aligned}
& \Gamma^* \nvdash \neg\psi \\
\text{iff}\quad & \Gamma^* \vdash \neg\forall x_1 \ldots \forall x_k\ \neg\psi \quad (\textit{if} \text{ part by the maximality of } \Gamma^*) \\
\text{iff}\quad & \Gamma^* \vdash \exists\psi \\
\text{iff}\quad & \text{for some } c_i \in \sigma\text{:} \quad \Gamma^* \vdash \psi(\vec{x}/\vec{c}) \\
\text{iff}\quad & \text{for some } c_i \in \sigma\text{:} \quad \mathfrak{A} \models \psi(\vec{x}/\vec{c}) \quad \text{(by the induction hypothesis)} \\
\text{iff}\quad & \mathfrak{A} \models \psi[e(\vec{x}/\vec{[c]})] \quad \text{(by the substitution theorem)} \\
\text{iff}\quad & \mathfrak{A} \models \exists x_1 \ldots \exists x_k\ \psi \\
\text{iff}\quad & \mathfrak{A} \not\models \neg\psi
\end{aligned}
$$

Case 4. $\varphi \equiv \psi_1 \wedge \psi_2$.

$$
\begin{aligned}
& \Gamma^* \vdash \psi_1 \wedge \psi_2 \\
\text{iff}\quad & \text{for all } c_i \in \sigma\text{:} \quad \Gamma^* \vdash \psi_1 \wedge \psi_2(\vec{x}/\vec{c}) \quad \text{(by Lemma 1)} \\
\text{iff}\quad & \text{for all } c_i \in \sigma\text{:} \quad \Gamma^* \vdash \psi_1(\vec{x}/\vec{c}) \quad \text{AND} \quad \Gamma^* \vdash \psi_2(\vec{x}/\vec{c}) \\
\text{iff}\quad & \text{for all } c_i \in \sigma\text{:} \quad \mathfrak{A} \models \psi_1(\vec{x}/\vec{c}) \quad \text{AND} \quad \mathfrak{A} \models \psi_2(\vec{x}/\vec{c}) \\
& \quad \text{(by the induction hypothesis)} \\
\text{iff}\quad & \mathfrak{A} \models \psi_1 \wedge \psi_2 \quad \text{(by the substitution theorem and the construction of } A)
\end{aligned}
$$

Case 5. Finally, $\varphi \equiv \exists y\ \psi$.

$$
\begin{aligned}
& \Gamma^* \vdash \exists y\ \psi \\
\text{iff}\quad & \text{for all } c_i \in \sigma \text{ and some } d \in \sigma\text{:} \quad \Gamma^* \vdash \psi(\vec{x}/\vec{c}, y/d) \quad \text{(by Lemma 1 and Henkin property)} \\
\text{iff}\quad & \text{for all } c_i \in \sigma \text{ and some } d \in \sigma\text{:} \quad \mathfrak{A} \models \psi(\vec{x}/\vec{c}, y/d) \quad \text{(by the induction hypothesis)} \\
\text{iff}\quad & \mathfrak{A} \models \exists y\ \psi \quad \text{(by the substitution theorem and the construction of } A)
\end{aligned}
$$

□