9/1. *Let Γ be a theorem which can represent every recursive function. Prove that for every pair of formulae $\Phi(x)$ and $\Psi(x)$ with one free variable, there exist closed formulae η and θ such that $\Gamma \vdash \eta \leftrightarrow \Phi(\ulcorner\theta\urcorner)$ and $\Gamma \vdash \theta \leftrightarrow \Psi(\ulcorner\eta\urcorner)$.*

Define the function $h : \mathbb{N}^2 \to \mathbb{N}$ as follows:

$$h(\alpha(\varphi), i) := \begin{cases} \alpha(\varphi(\Delta_i, \Delta_{\alpha(\varphi)})) & \text{, if } \varphi \text{ is a formula with two free variables} \\ 0 & \text{, otherwise} \end{cases}$$

Let the $h(x, y)$ function be represented by the $H(x, y, z)$ formula. Furthermore, define the formulae ξ_1, ξ_2, θ, η as follows:

$$\begin{aligned} \xi_1(x, y) &:\equiv \forall z(H(x, y, z) \rightarrow \Psi(z)) \\ \xi_2(x, y) &:\equiv \forall z(H(x, y, z) \rightarrow \Phi(z)) \\ \theta &:\equiv \xi_1(\ulcorner\xi_2\urcorner, \ulcorner\xi_1\urcorner) \equiv \forall z(H(\ulcorner\xi_2\urcorner, \ulcorner\xi_1\urcorner, z) \rightarrow \Psi(z)) \\ \eta &:\equiv \xi_2(\ulcorner\xi_1\urcorner, \ulcorner\xi_2\urcorner) \equiv \forall z(H(\ulcorner\xi_1\urcorner, \ulcorner\xi_2\urcorner, z) \rightarrow \Phi(z)) \end{aligned}$$

Then from the representation of h it follows that

$$\Gamma \vdash \forall z(H(\ulcorner\xi_1\urcorner, \ulcorner\xi_2\urcorner, z) \leftrightarrow z = \ulcorner\theta\urcorner) \tag{1a}$$

$$\Gamma \vdash \forall z(H(\ulcorner\xi_2\urcorner, \ulcorner\xi_1\urcorner, z) \leftrightarrow z = \ulcorner\eta\urcorner) \tag{1b}$$

The definition of θ makes the following formula an axiom:

$$\Gamma \vdash \theta \leftrightarrow \xi_1(\ulcorner\xi_2\urcorner, \ulcorner\xi_1\urcorner)$$

which is equivalent to the following:

$$\Gamma \vdash \theta \leftrightarrow \forall z(H(\ulcorner\xi_2\urcorner, \ulcorner\xi_1\urcorner, z) \rightarrow \Psi(z)) \tag{2a}$$

The definition of η makes the following formula an axiom:

$$\Gamma \vdash \eta \leftrightarrow \xi_2(\ulcorner\xi_1\urcorner, \ulcorner\xi_2\urcorner)$$

which is equivalent to:

$$\Gamma \vdash \eta \leftrightarrow \forall z(H(\ulcorner\xi_1\urcorner, \ulcorner\xi_2\urcorner, z) \rightarrow \Phi(z)) \tag{2b}$$

Because of (1a) and (2b) we have:

$$\Gamma \vdash \eta \leftrightarrow \Phi(\ulcorner\theta\urcorner)$$

Because of (1b) and (2a) we have:

$$\Gamma \vdash \theta \leftrightarrow \Psi(\ulcorner\eta\urcorner). \qquad \blacksquare$$